Math 564: Advance Analysis 1
Lecture 13

Cor. Let $f_n \in L^+$, then $\sum_{n\in\mathbb{N}} \int f_n d\mu = \int \sum_{n\in\mathbb{N}} f_n d\mu$

Proof.

First we show $\int (f_0 + f_1) d\mu = \int f_0 d\mu + \int f_1 d\mu$.
Let s_n, t_n be simple functions s.t. $0 \le s_n \nearrow f_0$ and $0 \le t_n \nearrow f_1$
then by the MCT, we have $\int s_n d\mu \to \int f_0$ and
$\int t_n d\mu \to \int f_1 d\mu$, as well as $\int (s_n + t_n) d\mu \to \int f_0 + f_1 d\mu$, so

$$\int (f_0 + f_1) d\mu = \lim_n \int (s_n + t_n) d\mu = \lim_n \left(\int s_n d\mu + \int t_n d\mu \right) = \int f_0 d\mu + \int f_1 d\mu.$$

For infinite sums, note that $\sum_{i<n} f_i \nearrow \sum_{i\in\mathbb{N}} f_i$ so by the MCT,

$$\int \sum_{i<n} f_i \, d\mu \nearrow \int \sum_{i\in\mathbb{N}} f_i \, d\mu$$

by linearity

$$\overset{\|}{\sum_{i<n}} \int f_i \, d\mu$$

$\square$

In particular, $\int$ is a linear non-neg. functional on L^+.

Cor. For $f \in L^+$, the map $B \mapsto \int_B f \, d\mu$ is a measure on $MEAS_\mu$, denoted by μ_f.

Proof. Let $B = \bigsqcup_{n\in\mathbb{N}} B_n$. Then $\mu_f(B) = \int_B f \, d\mu = \int f \cdot \mathbb{1}_B \, d\mu \overset{\text{by disjointness}}{=} \int \sum_n f \cdot \mathbb{1}_{B_n} d\mu$

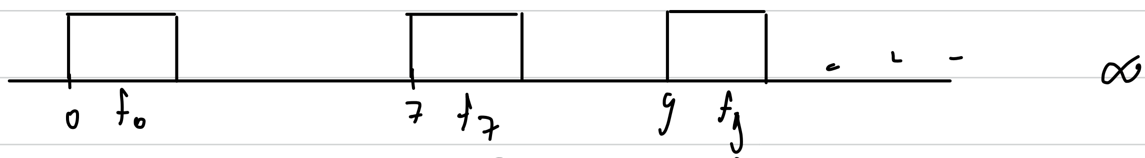
$$\overset{\text{MCT}}{=} \sum_n \int f \cdot \mathbb{1}_{B_n} d\mu = \sum_n \mu(B_n).$$

$\square$

We would like to understand when $f_n \to f$ ptwise gives $\int f_n \to \int f$.

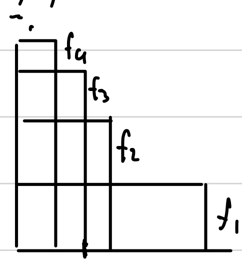
If $(f_n) \subseteq \mathcal{L}^+$ is not monotone, then weird things happen:

Examples. (a) $(X,\mu) := (\mathbb{R}, \lambda)$. Let $f_n := \mathbb{1}_{[n,n+1)} \to 0$ but $\int f_n \, d\lambda = 1$



(a') Let $f_n := \mathbb{1}_{[n,2n)} \to 0$ but $\int f_n \, d\lambda = n \to \infty$

(b) $(X,\mu) := ([0,1), \lambda)$. Let $f_n := \mathbb{1}_{(0,\frac{1}{n}]} \cdot n \to 0$ but $\int f_n d\lambda = 1$.



(b') Let $f_n := \mathbb{1}_{(0,\frac{1}{n})} \cdot n^2 \to 0$ but $\int f_n \, d\lambda = \frac{1}{n} \cdot n^2 = n \to \infty$.

Fatou's lemma. For any sq. $(f_n) \subseteq \mathcal{L}^+$, $\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n d\mu$.

Proof. $\liminf_n f_n = \lim_N \left(\inf_{n \geq N} f_n \right)$ and $\inf_{n \geq N} f_n$ increase so by the MCT, we have $\int \lim_{N\to\infty} (\inf_{n\geq N} f_n) \, d\mu = \lim_{N\to\infty} \int \inf_{n\geq N} f_n \, d\mu$

$\leq \lim_{N\to\infty} \inf_{n\geq N} \int f_n \, d\mu$ by the monotonicity of the integral. $\square$

Remark. Fatou $\Rightarrow$ MCT.

Integral of all functions. Call a μ-measurable function $f : X \to \overline{\mathbb{R}} := [-\infty, +\infty]$ *μ-integrable* if $\int f^+ d\mu, \int f^- d\mu < \infty$.

Equiv: $\int |f| \, d\mu < \infty$. In this case, we define

$$\int f \, d\mu := \int f^+ d\mu - \int f^- d\mu.$$

If $f: X \to \mathbb{C}$, then call it *μ-integrable* if $\operatorname{Re} f$ and $\operatorname{Im} f$ are both μ-integrable, and

$$\int f \, d\mu := \int \operatorname{Re} f \, d\mu + i \int \operatorname{Im} f \, d\mu.$$

Let $L^1(X,\mu)$ denote the set of all μ-integrable real (or complex) valued functions. For $f \in L^1 := L^1(X,\mu)$, denote:

$$\|f\|_1 := \int |f| \, d\mu,$$

and call it the *L^1-norm* of f.

Obs. $\|\cdot\|_1$ is a pseudo-norm on L^1:
(i) $\|f\|_1 = 0 \iff f = 0$ a.e. (this a.e. is why $\|\cdot\|_1$ isn't a norm)
(ii) $\|c \cdot f\|_1 = |c| \cdot \|f\|_1$ for all constants c.
(iii) $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$

Pf. $\int |f+g| \, d\mu \le \int (|f|+|g|) \, d\mu = \int |f| \, d\mu + \int |g| \, d\mu.$ $\square$
monotonicity of $\int$ and Δ-ineq on $\mathbb{R}$

Obs. L^1 is a vector space over $\mathbb{R}$ (resp., $\mathbb{C}$).

Thus, L^1 is a normed vector space. Every (pseudo) norm defines a (pseudo-) metric: $d_1(f,g) := \|f-g\|_1$. So it makes to talk about convergence in this metric/norm: write $f_n \to_{L^1} f$, read (f_n) *converges to f in L^1 norm*, if $\|f-f_n\|_1 \to 0$ as $n \to \infty$.

Let's again investigate when ptwise convergence gives L^1 convergence.

Obs. If $0 \le f_n \nearrow f$ then $f_n \to_{L^1} f$.
Proof. By MCT, $\int f \, d\mu = \lim \int f_n \, d\mu$, so $\lim_n \int (f-f_n) \, d\mu = 0$. $\square$

The examples above show that in each case where $\lim_n \int f_n > \int \liminf f_n$, the sequence (f_n) was not under a finite-area graph.
If we preclude this, then it works:

Dominated Convergence Theorem. Let $(f_n) \subseteq L^1$ and suppose $f_n \to f$ a.e.
If $|f_n| \le g$ for all n, for some $g \in L^1$, then $\int f_n \, d\mu \to \int f \, d\mu$.
In fact, $f_n \to_{L^1} f$.

Proof. Attempt 1: Apply Fatou to $|f_n|$, then $\int \liminf_n |f_n| \le \liminf_n \int |f_n|$
so $\int |f| \, d\mu \le \liminf_n \int f_n \, d\mu$.

<u>Attempt 2:</u> Other non-neg. function on the table are $g - f_n$ and $g + f_n$, so let's apply Fatou to them:

$$\cancel{\int g \, d\mu} - \int f \, d\mu = \int \lim (g - f_n) \, d\mu \le \liminf_n \int (g - f_n) \, d\mu = \cancel{\int g} - \limsup_n \int f_n \, d\mu$$

$$\cancel{\int g \, d\mu} + \int f \, d\mu = \int \lim (g + f_n) \, d\mu \le \liminf_n \int (g + f_n) \, d\mu = \cancel{\int g} + \liminf_n \int f_n \, d\mu$$

$$\limsup_n \int f_n \, d\mu \le \int f \, d\mu \le \liminf_n \int f_n \, d\mu.$$

Also, $|f_n| \le g \Rightarrow |f| \le g$ so $\|f\|_1 \le \|g\|_1 < \infty$, so $f \in L^1$.

The "in fact" part now follows from what we proved by applying it to $f_n - f$. Indeed, $|f_n - f| \le |f_n| + |f| \le 2g$ so $\int |f - f_n| \, d\mu \to \int 0 \, d\mu = 0$. $\square$

Prop. Simple functions are dense in L^1.

Proof. Let $f \in L^1$. This follows directly from the def of integral applied to f^+ and f^- separately. But one can also get it a sequence (s_n) of simple funct. s.t. $|s_n| \nearrow |f|$ and $s_n \to f$ ptwise hence then DCT applies and gives $s_n \to_{L^1} f$. $\square$

Def. Let (X, μ) be a measure space. We say that $\mathcal{E} \subseteq MEAS_\mu$ *generates* $MEAS_\mu$ *(mod μ-null)* if for every $A \in MEAS_\mu$ $\exists\, \tilde{A} \in \langle \mathcal{E} \rangle_\sigma$ s.t. $A =^\mu \tilde{A}$. We say that $MEAS_\mu$ is *ctbly generated* if there is a ctbl collection $\mathcal{E} \subseteq MEAS_\mu$ that is generating $MEAS_\mu$ (mod μ-null).

Examples. Lebesgue-measurable σ-alg, Bernoulli(p)-measurable σ-alg. are ctbly generated.

Prop. If (X, μ) is ctbly generated (i.e. $MEAS_\mu$ is ctbly gen mod μ-null), then $L^1(X, \mu)$ is separable.

Proof. HW.

We'll show later that $L^1(X, \mu)$ is also always complete.

Chebyshev's inequality. For $f \in L^1$ and $\alpha \in (0, \infty]$,

$$\mu\big(\overbrace{\{x \in X : |f(x)| \geq \alpha\}}^{A_\alpha}\big) \leq \frac{1}{\alpha} \cdot \|f\|_1 .$$

Proof. $\|f\|_1 = \int |f| \, d\mu \geq \int_{A_\alpha} |f| \, d\mu \geq \int_{A_\alpha} \alpha \, d\mu = \alpha \cdot \mu(A_\alpha)$. $\square$

Prop. Let $f \in L^1$ be extended real valued, i.e. $f : X \to [-\infty, \infty]$.

(a) $|f| < \infty$ a.e., i.e. $\{x \in X : f(x) = \pm\infty\}$ is null.

(b) $A := \{x \in X : f(x) \neq 0\}$ is σ-finite, i.e. $A = \bigcup_n A_n$ where $\mu(A_n) < \infty$ for all n.

Proof. (a) $\infty \cdot \mu(\underbrace{\{x \in X : |f(x)| = \infty\}}_{X_\infty}) \leq \|f\|_1 < \infty$, so $\mu(X_\infty) = 0$.

(b) Take $A_n := \{x \in X : |f(x)| \geq \frac{1}{n}\}$ then $A = \bigcup_n A_n$ and

by Chebyshev, $\mu(A_n) \leq \frac{1}{1/n} \cdot \|f\|_1 < \infty$. $\square$